

# Solovay's inaccessible over a weak set theory without choice

Haim Horowitz and Saharon Shelah<sup>1</sup>

## Abstract

We study the consistency strength of Lebesgue measurability for  $\Sigma_3^1$  sets over a weak set theory in a completely choiceless context. We establish a result analogous to the Solovay-Shelah theorem.

## Introduction

The following work is motivated by the results from [HwSh1067]. Assuming  $ZFC$  without large cardinals, we constructed a model of  $ZF$  where every set of reals is measurable (in a slightly weaker sense) with respect to the ideal derived from a Suslin ccc non-sweet creature forcing (a set of reals is measurable with respect to an ideal  $I$  if it equals a Borel set modulo  $I$ ). As the resulting model doesn't satisfy  $AC_\omega$ , the following questions arise:

1. Given a set theory  $T$  that doesn't prove  $AC_\omega$ , a Suslin ccc forcing notion  $\mathbb{Q}$  and an infinite cardinal  $\kappa$ , is  $T$  equiconsistent with  $T + \text{"Every set of reals is } I_{\mathbb{Q},\kappa}\text{-measurable"}$  (see [HzSh1067] for the definition of  $I_{\mathbb{Q},\kappa}$ )?
2. For  $T$ ,  $\mathbb{Q}$  and  $\kappa$  as above, is  $T + \text{"Every set of reals is } I_{\mathbb{Q},\kappa}\text{-measurable"}$  equiconsistent with  $T + AC_\omega + \text{"Every set of reals is } I_{\mathbb{Q},\kappa}\text{-measurable"}$ ?

Our main goal is to prove that both of the above questions have a negative answer when  $T = Z_*$  is a weak set theory that will be defined below,  $\mathbb{Q}$  is random real forcing and  $\kappa = \aleph_0$ . As  $Z_*$  doesn't include the Replacement scheme, we can't prove the existence of a limit uncountable cardinal in  $Z_*$ , so the point is that the role of the inaccessible cardinal in the context of  $ZFC$  is replaced by an uncountable limit cardinal in the context of  $Z_*C$ . Our proof will follow the old proofs of Solovay ([So]) and Shelah ([Sh176]) on the consistency strength of  $\Sigma_3^1$ -Lebesgue measurability, and our main goal is to show that similar arguments yield an analogous result in  $Z_*$ , where the inaccessible cardinal is replaced by an uncountable limit cardinal.

## Basic definitions

---

<sup>1</sup>Date: September 4, 2016

2000 Mathematics Subject Classification: 03E15, 03E35, 03E25, 03E30

Keywords: inaccessible cardinals, Lebesgue measurability, weak set theories, axiom of choice

Publication 1094 of the second author

Partially supported by European Research Council grant 338821

---

**Definition 1:** 1.  $Z^-$  is Zermelo set theory (i.e.  $ZF$  without replacement) without choice and without the powerset axiom (but with the separation scheme).

2.  $Z^- + \aleph_n$  is  $Z^- +$  "the cardinal  $\aleph_n$  exists".

3.  $Z_*$  is the theory that consists of the following axioms:

- a.  $Z^- + \aleph_1$ .
- b.  $P(\mathbb{N})$  exists.
- c.  $L_\alpha[z]$  exists for every ordinal  $\alpha$  and  $z \in \omega^\omega$ .
- d.  $\alpha + \omega$  exists for every ordinal  $\alpha$ .
- e. There is no greatest ordinal.

**Observation:** Recall that there exists a formula  $\phi(x)$  in the language of set theory such that:

- a. If  $\delta > \omega$  is a limit ordinal,  $a \in \omega^\omega$  and  $b = L_\delta[a]$ , then  $(b, \in) \models \phi(a)$ .
- b. If  $b$  is a transitive set,  $a \in \omega^\omega$  and  $(b, \in) \models \phi(a)$ , then there is a limit ordinal  $\delta > \omega$  such that  $b = L_\delta[a]$ .

Convention: From now on, our background theory is  $Z_*$ , we do not assume  $AC_\omega$  and by "Borel sets" we refer only to sets of reals having a Borel code.

We shall now define several versions of Lebesgue measurability and the null ideal (not that the different versions are not equivalent without choice).

**Definition 2:** 1. A set  $X \subseteq \mathbb{R}$  is 1-null if there exists a Borel set  $B$  such that  $X \subseteq B$  and  $\mu(B) = 0$ .

2. A set  $X \subseteq \mathbb{R}$  is 2-null if for every  $n < \omega$  there exists a Borel set  $B_n$  such that  $X \subseteq B_n$  and  $\mu(B_n) < \frac{1}{n+1}$ .

**Remark:** 1-null implies 2-null, and the definitions are equivalent under  $AC_\omega$ .

**Definition 3:** A set  $X \subseteq \mathbb{R}$  is  $i$ -measurable ( $i = 1, 2, 3$ ) if:

$i = 1$  : There exists a Borel set  $B$  such that  $X \Delta B$  is 1-null.

$i = 2$  : There exists a Borel set  $B$  such that  $X \Delta B$  is 2-null.

$i = 3$  : For every  $n < \omega$ , there exist Borel sets  $B_1$  and  $B_2$  such that  $X \Delta B_1 \subseteq B_2$  and  $\mu(B_2) < \frac{1}{n+1}$ .

It's easy to see that  $i$ -measurability implies  $j$ -measurability for  $i < j$ .

**Claim and notation 4:** 1.  $X$  is 3-measurable iff  $X$  has the same inner measure and outer measure, which will be denoted by  $\mu^*$ .

2.  $X \subseteq \mathbb{R}$  is 2-null iff  $\mu^*(X) = 0$ .  $\square$

## A lower bound on the consistency strength

**Claim 5:** a. Suppose that  $V \models Z_*$ . The following version of Fubini's theorem holds:

If  $A \subseteq \mathbb{R}$  is not 2-null and  $\leq$  is a prewellordering of  $A$  such that every initial segment (i.e.  $\{y : y \leq x\}$ ) is 2-null, then there exists a set which is not 3-measurable.

2. If in addition  $\leq$  is  $\Sigma_2^1$ , then there exists a non-3-measurable  $\Sigma_3^1$  set. In particular, throughout the following proof, at least one of the following  $\Sigma_3^1$  sets is not 3-measurable:  $C_{B,a}$ ,  $B_{n,i}$ ,  $A$ ,  $B_0$ ,  $B_1$  and  $B_2$ .

**Proof:** Suppose that all sets are 3-measurable and we shall derive a contradiction. For every Borel set  $B \subseteq [0, 1] \times [0, 1]$  and  $a \in [0, 1]$ , define  $C_{B,a} := \{s_1 \in [0, 1] : a \leq \mu(\{s_2 : (s_1, s_2) \in B\})\}$ , and similarly, define  $C_{a,B} := \{s_2 \in [0, 1] : a \leq \mu(\{s_1 : (s_1, s_2) \in B\})\}$ .

Subclaim 1: Let  $B \subseteq [0, 1] \times [0, 1]$  be a Borel set coded by  $r$ , and let  $a, b, r_1 \in \omega^\omega$  such that  $r, a, b \in L[r_1]$ , if  $L[r_1] \models \mu(C_{B,a}) = b$  then  $V \models \mu(C_{B,a}) = b$ .

Proof: In  $L[r]$  there is a sequence  $(U_n, S_n : n < \omega)$  such that the sets  $U_n \subseteq [0, 1]$  are open, the sets  $S_n \subseteq [0, 1]$  are closed,  $S_n \subseteq C_{B,a} \subseteq U_n$  and  $L[r] \models \mu(U_n \setminus S_n) < \frac{1}{n}$ . We shall prove that  $\mu(U_n)^V = \mu(U_n)^{L[r]}$ ,  $\mu(S_n)^V = \mu(S_n)^{L[r]}$  and  $V \models \mu(C_{B,a}) = \mu(C_{B,a})^{L[r]}$  for every  $n < \omega$ .

We shall work in  $L[r]$  and assume wlog that  $S_n \subseteq S_{n+1}$  for every  $n < \omega$ . Define  $R$  as the set of triples  $(n, s_1, S)$  such that:

1.  $n < \omega$ ,  $s_1 \in S_n \subseteq C_{B,a}$ .
2.  $S \subseteq [0, 1]$  is closed and  $\mu(S) = a - \frac{1}{n}$ .
3.  $s_1 \times S \subseteq B$ .

Let  $X = \omega \times [0, 1]$  and  $Y = \{S : S \subseteq [0, 1] \text{ is closed}\}$ , then  $X$  and  $Y$  are Polish spaces and  $R \subseteq X \times Y$  is a  $\Pi_1^1$  relation. By  $\Pi_1^1$ -uniformization, there is a function  $F \subseteq R$  with a  $\Pi_1^1$  graph such that  $\text{Dom}(F) = \text{Dom}(R)$ . By absoluteness, the same is true for  $(R, F)$  in  $V$ . Now, if  $s_1 \in S_n$  then  $s_1 \in S_m$  for every  $m > n$  and  $\{F(m, s_1) : n \leq m\}$  witnesses that  $s_1 \in C_{B,a}$ . Therefore,  $V \models \sup\{\mu(S_n) : n < \omega\} \leq \mu(C_{B,a})$ . Similarly we can show that  $V \models \mu(C_{B,a}) \leq \inf\{\mu(U_n) : n < \omega\}$ .

Subclaim 2: If  $B \subseteq [0, 1] \times [0, 1]$  is Borel and  $a, b \in [0, 1]$ , then  $\mu(C_{B,a}) = b \rightarrow b \leq \frac{\mu(B)}{a}$ .

Proof: Let  $r_1$  be a real such that  $a, b$  and the definition of  $B$  (hence of  $C_{B,a}$ ) are in  $L[r_1]$ . As the conclusion holds in  $L[r_1]$ , it follows from the previous claim that it holds in  $V$  as well.

Subclaim 3: Assume  $Z_*$ . Fubini's theorem holds for Borel and analytic sets in the following sense: If  $B \subseteq [0, 1] \times [0, 1]$  is Borel/analytic and  $f_l : [0, 1] \rightarrow [0, 1]$  ( $l = 1, 2$ ) are defined by  $f_l(s_l) := \mu(\{s_{3-l} : (s_1, s_2) \in B\})$ , then  $\mu(B) = \int f_1(s_1) ds_1 = \int f_2(s_2) ds_2$ .

Proof: Let  $r$  be a real such that the definition of  $B$  is in  $L[r]$ , and we shall continue the proof as usual in  $L[r]$ . The only point that we have to show is that the above integrals are well-defined and computed in the same way in  $L[r]$  and  $V$ . For every  $n > 1$  and  $i \leq n$ , let  $B_{n,i} := \{s_1 : \mu(B_{s_1}^1) := \mu(\{s_2 : (s_1, s_2) \in B\}) \in [\frac{i}{n}, \frac{i+1}{n}]\}$ .

$(B_{n,i} : i \leq n)$  is a partition of  $[0, 1]$  for every  $n$ . For every  $n$ , choose a sequence  $(S_{n,i}, U_{n,i} : i \leq n)$  in  $L[a]$  such that  $S_{n,i} \subseteq B_{n,i} \subseteq U_{n,i}$ ,  $S_{n,i}$  is closed,  $U_{n,i}$  is open and  $\mu(U_{n,i} \setminus S_{n,i}) < \frac{1}{2^n}$ . Let  $R_1$  be the set of sequences  $(n, i, s_1, S)$  such that:

1.  $n > 1$  and  $i \leq n$ .
2.  $s_1 \in S_{n,i}$ .
3.  $S \subseteq [0, 1]$  is closed and  $\frac{1}{n} - \frac{1}{2^n} \leq \mu(S)$ .
4.  $S \subseteq B_{s_1}^1$ .

Let  $X := \{(n, i, s_1) : i \leq n, n > 1, s_1 \in S_{n,i}\}$  and  $Y$  be the set of closed subsets of  $[0, 1]$ . As before, by  $\Pi_1^1$ -uniformization, there is a  $\Pi_1^1$ -function  $F_1 \subseteq R$  such that for every  $(n, i, s_1)$ , if there exists  $S$  such that  $(n, i, s_1, S) \in R_1$ , then  $(n, i, s_1, F_1(n, i, s_1)) \in R_1$ . By absoluteness, the same is true in  $V$ . Similarly, define  $R_2$  as the set of sequences  $(n, i, s_1, U)$  such that:

1.  $n > 1$  and  $i \leq n$ .
2.  $s_1 \in U_{n,i}$ .
3.  $U \subseteq [0, 1]$  is open and  $\mu(U) < \frac{i+1}{n} + \frac{1}{2^n}$ .
4.  $B_{s_1}^1 \subseteq U$ .

As before, there is  $\Pi_1^1$  choice function  $F_2$  for the relation  $R_2$ .  $F_1$  and  $F_2$  witness that the above integrals are well-defined and have the same value in  $L[r]$  and  $V$ .

Subclaim 4: If  $A \subseteq [0, 1]$  and  $B = A \times A$ , then  $\mu^*(B) = \mu^*(A)^2$ .

Proof: In one direction, let  $a = \mu^*(A)$  and  $\epsilon > 0$ . There is a Borel set  $A^*$  such that  $A \subseteq A^* \subseteq [0, 1]$  and  $\mu(A^*) \leq \mu(A) + \epsilon$ . Let  $B^* = A^* \times A^*$ , then  $\mu^*(B) \leq \mu^*(A^* \times A^*) = \mu(A^*)^2 \leq (a + \epsilon)^2$ . Therefore,  $\mu^*(B) \leq \mu^*(A)^2$ .

In the other direction, let  $a = \mu^*(A)$ ,  $b = \mu^*(B)$  and  $\epsilon > 0$ . There are Borel sets  $A^*$  and  $B^*$  such that  $A \subseteq A^*$ ,  $B \subseteq B^*$ ,  $\mu(A^*) \leq \mu^*(A) + \epsilon$  and  $\mu(B^*) \leq \mu^*(B) + \epsilon$ . Without loss of generality,  $B^* \subseteq A^* \times A^*$ . If  $s_1 \in A$  then  $a \leq \mu(\{s_2 : (s_1, s_2) \in B^*\})$ , therefore  $a = \mu^*(A) \leq \mu^*(C_{B,a}) \leq \mu^*(A^*) < a + \epsilon$ . By Fubini's theorem for Borel sets, it follows that  $a^2 \leq \mu^*(B^*)$ . Therefore,  $\mu^*(A)^2 - \epsilon = a^2 - \epsilon \leq \mu^*(B^*) - \epsilon \leq \mu^*(B)$ , so  $\mu^*(B) = \mu^*(A)^2$  as required.

We are now ready to complete the proof of claim 5.

Without loss of generality  $A \subseteq [0, 1]$ . We now define the following sets:

1.  $B_0 = B = A \times A$
2.  $B_1 = \{(x, y) \in B : x \leq y\}$
3.  $B_2 = \{(x, y) \in B : y \leq x\}$

Suppose that each of the sets  $A$ ,  $B_0$ ,  $B_1$  and  $B_2$  are 3-measurable and we shall derive a contradiction. Choose  $\epsilon_1, \epsilon_2 \in (0, 1)$  such that  $\epsilon_1 < \epsilon_2^2$  and  $\epsilon_2 < \frac{a^2}{6}$  (recall that  $A$  is not 2-null by our assumption). As  $A$  is 3-measurable, there are Borel sets  $A^*$

and  $A^{**}$  such that  $A \Delta A^* \subseteq A^{**}$  and  $\mu(A^{**}) < \epsilon_1$ . Similarly, there are Borel sets  $B_l^*$  and  $B_l^{**}$  ( $l = 1, 2$ ) such that  $B_l \Delta B_l^* \subseteq B_l^{**}$  and  $\mu(B_l^{**}) < \epsilon_1$ . We shall prove that  $\mu(B_l^* \cup B_l^{**}) < 3\epsilon_2$ . Together we obtain the following:

$$a^2 = \mu^*(A \times A) = \mu^*(B_1 \cup B_2) \leq \mu^*(B_1^* \cup B_1^{**} \cup B_2^* \cup B_2^{**}) \leq \mu(B_1^* \cup B_1^{**}) + \mu(B_2^* \cup B_2^{**}) < 3\epsilon_2 + 3\epsilon_2 < a_2, \text{ a contradiction.}$$

By a previous subclaim,  $\mu^*(C_{B_2^{**}, \epsilon_2})^V \leq \mu^*(C_{B_2^{**}, \epsilon_2})^{L[a]} \leq \frac{\mu^*(B_2^{**})}{\epsilon_2} < \epsilon_2$  (where  $a$  is as in the subclaim). Let  $C_2 := C_{B_2^{**}, \epsilon_2}$  and let  $B_2' := B_2^* \cap ([0, 1] \setminus C_2 \times [0, 1])$ . The following inequalities hold:

1.  $\mu^*(B_2^*) \leq \mu^*(B_2^* \cap (C_2 \times [0, 1])) + \mu^*(B_2^* \cap ([0, 1] \setminus C_2 \times [0, 1]))$
2.  $\mu^*(B_2^* \cap (C_2 \times [0, 1])) \leq \mu^*(C_2 \times [0, 1]) \leq \mu^*(C_2) < \epsilon_2$

Therefore, it suffices to show that  $\mu^*(B_2^* \cap ([0, 1] \setminus C_2 \times [0, 1])) \leq \epsilon_2$ . Given  $s_2 \in [0, 1] \setminus C_2$ , the following holds:  $\mu^*(\{s_2 : (s_1, s_2) \in B_2^*\}) \leq \mu^*(\{s_2 : (s_1, s_2) \in B_2\}) + \mu^*(\{s_2 : (s_1, s_2) \in B_2^{**}\}) \leq 0 + \epsilon_2$  where the last inequality follows by the choice of  $s_1$ , the definition of  $B_2$  and the theorem's assumption. By Fubini's theorem, the desired conclusion follows.

The proof for  $l = 1$  is similar, where  $C_{B_2^{**}, \epsilon_2}$  is replaced by  $C_{\epsilon_2, B_1^{**}}$  and the rest of the arguments are changed accordingly.  $\square$

**Theorem 6:** Assume  $Z_*$ .

1. If every  $\Sigma_3^1$  set of reals is 3-measurable, then  $\aleph_1^{L[x]} < \aleph_1$  for every  $x \in 2^\omega$ , hence  $\aleph_1$  is a limit cardinal in  $L$ .
2. If in addition  $AC_\omega$  holds, then  $\aleph_1$  is inaccessible in  $L$ .

**Proof:** We follow a similar argument as in [Sh176]. Assume towards contradiction that  $\aleph_1^{L[x_*]} = \aleph_1$  for some  $x_* \in 2^\omega$ . For every  $x \in 2^\omega$ , let  $(\mathbf{B}_{x,i} : i < i^*)$  list all of the Borel null subsets of  $2^\omega$  (i.e. their Borel codes, recalling that " $\mu(A) = 0$ " is absolute) in  $L[x_*, x]$  (we can do it uniformly in  $(x, x_*)$ ). Denote  $\mathbf{B}_{x,i}^* = \mathbf{B}_{x,i}^V$  and  $\mathbf{B}_{x, < i}^* = \bigcup_{j < i} \mathbf{B}_{x,j}^*$ . Let  $\mathbf{B}_x^* = \bigcup_{i < i^*} \mathbf{B}_{x,i}^*$ .

**Case I:** There exists  $x_{**} \in 2^\omega$  such that  $B_{x_{**}}^*$  is not 2-null.

Work in  $V$ : Denote  $\mathbf{B} = \mathbf{B}_{x_{**}}^*$  and define the following prewellordering on  $\mathbf{B}$ :  $x \leq y$  iff for every  $i$ ,  $y \in \mathbf{B}_{x_{**}, < i}^* \rightarrow x \in \mathbf{B}_{x_{**}, < i}^*$ .

Clearly, every initial segment of  $(\mathbf{B}, \leq)$  has the form  $\mathbf{B}_{x_{**}, < i}^*$ , and hence is 2-null. As  $B$  is not 2-null, it follows by claim 5 that there exists a non-3-measurable  $\Sigma_2^1$ -set, a contradiction.

**Case II:**  $B_x^*$  is 2-null for every  $x \in 2^\omega$ .

We shall first describe the original stages of the proof in [Sh176], then we shall describe how to modify the original proof in order to obtain the desired theorem. The new changes and arguments will be presented in this section, while the proofs from [Sh176] will appear in the appendix.

---

**Outline of [Sh176]:**

We fix a rapidly increasing sequence  $(\mu(k) : k < \omega)$  of natural numbers, say,  $\mu(k) = 2^{2^{2^{2^{176k}}}}$ .

**Step I (existence of a poor man generic tree):** Suppose that  $B \subseteq 2^\omega$  has measure zero, then there are perfect trees  $T_0, T_1 \subseteq 2^{<\omega}$ , functions  $m_l : T_l \rightarrow \mathbb{Q}$  and natural numbers  $n(k)$  ( $k < \omega$ ) such that  $\lim(T_l) \cap B = \emptyset$ ,  $m_l(\eta) = \mu(\lim(T_l) \cap (2^\omega)^{[\eta \leq]})$  and:

A) 1.  $m_0(<>) = \frac{1}{2}$  and for every  $\eta \in T_0$ ,  $\mu(\lim(T_0) \cap (2^\omega)^{[\eta \leq]})$  has the form  $\frac{k}{4^{lg(\eta)+1}}$  for  $0 \leq k \leq 4^{lg(\eta)+1}$ , and  $k \neq 0$  iff  $\eta \in T_0$ .

A) 2.  $m_1(<>) = \frac{1}{2}$ , and for every  $\eta \in T_1$ , if  $lg(\eta) \leq n(k)$  then  $\mu(\lim(T_1) \cap (2^\omega)^{[\eta \leq]}) \in \{\frac{1}{4^{n(k)+1}} : 9 < l < 4^{n(k)+1}\}$ .

B) For every  $\eta \in 2^{n(k)} \cap T_1$ ,  $2^{n(k)}(1 - \frac{1}{\mu(k)}) < \mu(\lim(T_1) \cap (2^\omega)^{[\eta \leq]})$ .

**Step II:** Definitions of finite and full systems (see definitions 1-4 in the appendix).

**Step III:** Showing that the family of finite systems satisfies ccc (claim 5 in the appendix).

**Step IV:** Forcing with finite systems over  $L[x_*]$  to get a full system in  $L[x_*]$ . As the existence of a full system is equivalent to the existence of a model to a  $\mathcal{L}_{\omega_1, \omega}(Q)$  sentence, this is sufficient by absoluteness and Keisler's completeness theorem.

**Step V:** We use the full system in order to define two  $\Sigma_3^1$  sets of reals (those are the red and the green sets in [Sh176]), which will turn out to be non-measurable.

**Step VI:** Showing that the green and red sets are disjoint, are not null and have outer measure 1, arriving at a contradiction.

**Back to the proof of theorem 6:**

We shall describe how each of the above steps should be modified in order to obtain the proof of our theorem.

**Step I: Claim:** The claim in step I of [Sh176] holds when  $B$  is a Borel set of measure (say)  $< \frac{1}{1000}$ . This will be used in order to show that the red and green sets are not 2-null (this is step VI).

**Proof:** Let  $r$  be a real that codes  $B$ . The proof is as in [Sh176], where now we work in  $L[r]$ . Observe that the tree  $T$  constructed there satisfies  $\lim(T) \cap A = \emptyset$  where  $A$  is an open set of measure  $< \frac{1}{1000}$  containing  $B$  (and the construction depends only on  $A$ ).

**Steps II-III:** No change is needed.

**Step IV:** Assuming  $Z^*$  we can prove Keisler's completeness theorem as well as the forcing theorem in  $L[r]$  for every  $r$  (see the discussion on forcing over models of  $Z_*$  in the end of this section). Therefore we can repeat the argument in the original Step IV.

**Step V:** No change.

**Step VI:** We shall freely use the notation and definitions from [Sh176] (see definition 7 and claims 8-11 in the appendix).

**Claim A:** The formulas  $\phi_{rd}$  and  $\phi_{gr}$  are contradictory.

**Proof:** Suppose that  $x$  satisfies both formulas. By definition 7 in the appendix, there is a poor man generic tree over  $L[x_*]$  denoted by  $T_0^{rd}$  and a poor man generic tree over  $L[x_*, T_0^{rd}]$  denoted by  $T_1^{rd}$  witnessing  $\phi_{rd}(x)$ . Repeating the proof of claim 9 in the appendix, in  $L[x_*, x, T_0^{rd}, T_1^{rd}]$  there is a partition  $\bar{A}^{rd} = \bar{A}^{rd}(x) = (A_n^{rd} : n < \omega)$  of  $\omega_1$  to countably many homogeneously red sets. Similarly, as  $x$  satisfies  $\phi_{gr}$ , in  $L[x_*, x, T_0^{rd}, T_1^{rd}, T_0^{gr}, T_1^{gr}]$  there is a partition  $\bar{A}^{gr} = \bar{A}^{gr}(x) = (A_n^{gr} : n < \omega)$  of  $\omega_1$  to countably many homogeneously green sets. As  $\omega_1$  is regular in  $L[x_*, x, T_0^{rd}, T_1^{rd}, T_0^{gr}, T_1^{gr}]$ , we get a contradiction.

**Claim B:** The formulas  $\phi'_{rd}$  and  $\phi'_{gr}$  are contradictory.

**Proof:** Suppose that  $\phi'_{rd}(z) \wedge \phi'_{gr}(z)$ , then for some  $x, y$  and natural  $n^*$  we have  $\phi_{rd}(x) \wedge \phi_{gr}(y)$  and  $\{n : x(n) \neq y(n)\} \subseteq \{0, \dots, n^*\}$ . Let  $\bar{A}^{rd}(x)$  and  $\bar{A}^{gr}(y)$  be as in the previous proof, and for every  $n, m < \omega$  let  $B_{n,m} = A_n^{rd} \cap A_m^{gr}$ . For some  $n, m$ ,  $B_{n,m}$  is infinite. Let  $\alpha_k$  be the  $k$ th element of  $B_{n,m}$ . Recalling that  $i_1 < i_2 < i_3 \rightarrow h(i_1, i_2) \neq h(i_2, i_3)$ , then for some  $k$  and  $j$  we have  $h(\alpha_k, \alpha_j) > n^*$ . Therefore  $red = x(h(\alpha_k, \alpha_j)) = y(h(\alpha_k, \alpha_j)) = green$  (recalling that  $\alpha_k, \alpha_j \in A_n^{rd} \cap A_m^{gr}$ ), which is a contradiction.

**Claim C:**  $A_{rd} = \{x : \phi_{rd}(x)\}$  and  $A_{gr} = \{x : \phi_{gr}(x)\}$  are not of measure zero.

**Proof:** This is the same argument as in claim 10 in the appendix, the only difference is that instead of taking a  $G_\delta$  set of measure zero covering  $A_{rd}$ , we take for every  $0 < \epsilon$  a Borel set of measure  $< \epsilon$  covering  $A_{rd}$ . By the modified construction of the poor man generic tree, we continue as in the original proof.

**Claim D:**  $A_{rd}$  is not 3-measurable.

**Proof:** As in [Sh176] (claim 11 in the appendix).  $\square$

## An upper bound on consistency strength (following Levy)

Historical remark: While Solovay's proof used the collapse of an inaccessible cardinal (which results in a model of  $DC$ ), our proof follows an older argument of Levy that used the collapse of a limit uncountable cardinal.

**Theorem 7:**  $A \rightarrow B$  where:

- A) 1.  $V \models Z^*C$ .
2.  $V = L$ .
3.  $\lambda$  is a limit cardinal  $> |P(\mathbb{N})|$  such that  $\mu < \lambda \rightarrow 2^\mu < \lambda$ .
4.  $\mathbb{P} = \Pi\{\mathbb{P}_{\mu,n} : \mu < \lambda, n < \omega\}$  is a finite support product such that  $\mathbb{P}_{\mu,n} = Col(\omega, \mu)$ .
5.  $G \subseteq \mathbb{P}$  is generic,  $\eta_{\mu,n} = \eta_{\mu,n}[G] : \omega \rightarrow \mu$  is the generic of  $\mathbb{P}_{\mu,n}$ .

---

6. In  $V[G]$  we define  $V_1 = V[\{\eta_{\mu,n} : \mu < \lambda, n < \omega\}]$ , i.e. the class of sets in  $V[G]$  hereditarily definable from parameters in  $V$  and a finite number of members of  $\{\eta_{\mu,n} : \mu < \lambda, n < \omega\}$ .

B) 1.  $V_1 \models Z^*$ .

2.  $V_1 \models \aleph_1 = \lambda$ .

3. If  $\lambda$  is singular in  $V$  then  $V_1 \models cf(\lambda) = \aleph_0$ .

4. If  $\lambda$  is regular in  $V$  then  $V_1 \models cf(\lambda) = \aleph_1$ .

5. The following claim holds in  $V_1$ : If (a)+(b)+(c) hold then (d) holds where:

a.  $\mathbb{Q}$  is a definition of a forcing notion (with elements which are either reals or belong to  $H(\aleph_1)$ ) with parameters in  $V_1$  satisfying c.c.c., such that  $\mathbb{Q}$  is absolute enough in the following sense: There is  $\bar{t}_* = ((\mu_i, n_i) : i < n(*))$  such that  $\mathbb{Q}$  is definable using  $\bar{\eta}_{t_*} = \{\eta_{\mu_i, n_i} : i < n(*)\}$  and parameters from  $V$ , and if  $\bar{t} = ((\mu_l, n_l) : l < n)$  then  $\mathbb{Q}^{V[\bar{\eta}_{t_*}, \bar{t}]} \leq \mathbb{Q}^{V_1}$ .

b.1.  $\eta$  is a  $\mathbb{Q}$ -name of a real, i.e. a sequence of  $\aleph_0$  antichains given in  $V[\bar{\eta}_{t_*}]$ .

b.2. The generic set can be constructed from  $\eta$  in a Borel way.

c. The ideal  $I = I_{(\mathbb{Q}, \eta), \aleph_0}$  (see [HwSh1067]) satisfies:  $\bar{t}_* \leq \bar{t}_1 \leq \bar{t}_2 \rightarrow I^{V[\bar{\eta}_{t_1}]} = P(P(\mathbb{N}))^{V[\bar{\eta}_{t_1}]} \cap I^{V[\bar{\eta}_{t_2}]}$ .

Remark: Note that  $P(\mathbb{N})^{V_1} = \cup \{P(\mathbb{N})^{V[\bar{\eta}_{t_1}]} : \bar{t}_1 \text{ has the form } ((\mu_i, n_i) : i < n)\}$ .

d. Every  $X \subseteq \omega^\omega$  equals a Borel set modulo  $I$ .

We shall first outline Solovay's original proof from [So], then we shall describe how to similarly prove the above theorem.

### An outline of Solovay's proof (for random real forcing)

**Step I:** Let  $G \subseteq \text{Coll}(\omega, < \kappa)$  be generic where  $\kappa$  is inaccessible and let  $x \in V[G] \cap \text{Ord}^\omega$ , then there exists a generic  $H \subseteq \text{Coll}(\omega, < \kappa)$  such that  $V[G] = V[x][H]$ .

**Step II:** For every formula  $\phi$  there is a formula  $\phi^*$  such that for every  $x \in V[G] \cap \text{Ord}^\omega$ ,  $V[G] \models \phi(x)$  iff  $V[x] \models \phi^*(x)$ .

**Step III:** In  $V[G]$ ,  $\omega^\omega \cap V[a]$  is countable for every  $a \in \text{Ord}^\omega$ .

**Step IV:** For every  $a \in \omega^\omega$ ,  $\{x \in \omega^\omega : x \text{ is not } (\mathbb{Q}, \eta)\text{-generic over } V[a]\} \in I$ , where  $\mathbb{Q}$  is random real forcing and  $\eta$  is the name for the generic.

**Step V:** Given a maximal antichain  $J \in V[a]$  of closed sets deciding  $\phi^*(a, \eta)$  (where  $\eta$  is the name for the random real), we define the desired Borel set as union of members of  $J$  forcing  $\phi^*(a, \eta)$ .



---

**Proof of theorem 7:** Suppose that  $A \subseteq \omega^\omega$  is definable using  $\bar{\eta}_t$  for  $\bar{t} = ((\mu_i, n_i) : i < n)$ . As before, we shall indicate how to modify Solovay's original proof for our purpose.

**Step I:** Our aim is to prove a result similar to Step I above, where the real parameter belongs to  $V_1$ . Suppose that  $a \in V_1$  is a real (so  $a = \underset{\sim}{a}[G]$  for some  $\mathbb{P}$ -name  $\underset{\sim}{a}$ ), then  $a$  is definable by a formula  $\phi$  from a finite number of  $\eta_{\mu_i, n_i}$ 's, say  $\{\eta_{\mu_i, n_i} : i < i(*)\}$ . In order to prove that  $a \in V[\{\eta_{\mu_i, n_i} : i < i(*)\}]$ , it's enough to show that:

Claim: If  $p \in \mathbb{P}$  and  $p \Vdash \underset{\sim}{a}(n) = k$  then  $p \restriction \prod_{i < i(*)} \mathbb{P}_{\mu_i, n_i} \Vdash \underset{\sim}{a}(n) = k$ .

Proof: Suppose towards contradiction that  $p \restriction \prod_{i < i(*)} \mathbb{P}_{\mu_i, n_i} \leq q$  forces a different value for  $\underset{\sim}{a}(n)$ . Let  $\pi$  be an automorphism of  $\mathbb{P}$  over  $\prod_{i < i(*)} \mathbb{P}_{\mu_i, n_i}$  such that  $\pi(p)$  is compatible with  $q$  (just switch the relevant coordinates), then  $\pi(p) \Vdash \underset{\sim}{a}(n) = k$ , a contradiction.

In order to complete this step, we shall prove the following claim:

Claim: If  $\mathbb{Q} \leq \prod_{i < i(*)} \mathbb{P}_{\mu_i, n_i}$  then there is an isomorphism of  $RO(\mathbb{P})$  onto  $RO(\mathbb{Q} \times \mathbb{P})$  that is the identity over  $RO(\mathbb{Q})$ .

Proof: Let  $\kappa > \mu > \aleph_1 + \max\{\mu_i : i < i(*)\}$ . As we assume that  $V = L$  (so in particular we have GCH), the usual proof works.

Conclusion: If  $G \subseteq \mathbb{P}$  is generic over  $V$  and  $a \in (\omega^\omega)^{V_1}$ , then there is a generic  $H \subseteq \mathbb{P}$  such that  $V[G] = V[a][H]$ .

Proof: By the above claims,  $a \in V[\{\eta_{\mu_i, n_i} : i < i(*)\}]$  for an appropriate finite set of  $\eta_{\mu_i, n_i}$ 's. Let  $B_a$  be the complete subalgebra generated by  $a$ , then by the previous claim  $B_a \times \mathbb{P}$  is isomorphic to  $\mathbb{P}$  (over  $B_a$ ) and the claim follows.

**Steps II:** Same as in Solovay's proof.

**Step III:** Suppose that  $G' \subseteq \prod_{i < i(*)} \mathbb{P}_{\mu_i, n_i}$  is generic. We shall use the fact that the ideal  $I$  is generated by sets which are disjoint to some  $B_N$  such that  $N \subseteq H(\aleph_1)^{L[G']}$  where:

1.  $N$  is transitive and  $||N|| = \aleph_0$ .
2.  $B_N = \{\underset{\sim}{\eta}[H] : H \text{ is } \mathbb{Q}^{L[G']} \cap N\text{-generic over } N\}$ .

Work in  $V_1$ : Let  $X$  be the set of  $\nu \in \omega^\omega$  (in  $V_1$ ) such that  $\nu$  is not  $(N, \mathbb{Q}, \underset{\sim}{\eta})$ -generic where  $N = (H^{V[G']}(\aleph_1), \in)$ . As  $N$  is countable in  $V_1$  (recall that  $\lambda$  is strong limit) and  $\nu \in (\omega^\omega)^{V_1}$  is generic over  $N$  iff it's generic over  $V[G']$ , it follows by the definition of  $I$  that  $X \in I$ .

**Step IV:** Suppose that  $A \subseteq \omega^\omega$  is definable by  $\phi(\bar{\eta}_t, x)$  where  $\bar{\eta}_t \in L[G']$  and  $G' \subseteq \prod_{i < n} \mathbb{P}_{\mu_i, n_i}$  is the generic set obtained by the restriction of  $G$  to  $\prod_{i < n} \mathbb{P}_{\mu_i, n_i}$ . Let  $\{p_n : n < \omega\} \subseteq \mathbb{Q}^{L[G']}$  be a maximal antichain and  $(\underset{\sim}{t}_n : n < \omega)$  a sequence of names

of truth values such that  $p_n \Vdash \eta \in A$  iff  $\mathbf{t}_n = \text{true}$  (such sequences exist by step II). Let  $\bar{p} = (\bar{p}^i : i < \omega)$  enumerate all maximal antichains in  $H(\aleph_1)^{L[G']}$  (so each  $\bar{p}^i$  is of the form  $\bar{p}^i = (p_n^i : n < \omega)$ ).

By our assumption, given a generic real  $\eta$  we can define the set  $G_\eta$  in a Borel way such that:

(\*)  $G_\eta$  is generic over  $H(\aleph_1)^{L[G']}$  and  $\eta[G_\eta] = \eta$ .

Now let  $B := \{\eta : G_\eta \text{ is well-defined, satisfies (*) above and for some } n, p_n \in G_\eta \wedge \mathbf{t}_n[G_\eta] = \text{true}\}$ . Denote by  $B_n$  the set of  $\eta \in B$  such that " $\eta \in B$ " is witnessed by  $n$ .

$B$  is Borel by our assumptions on the forcing. Therefore it's enough to prove that  $A = B \bmod I$ .

Let  $\eta \in \omega^\omega$  (in  $V_1$ ), by step III it's enough to show that if  $\eta$  is generic over  $H(\aleph_1)^{L[G']}$  (and hence  $\eta = \eta[G_\eta]$  for  $G_\eta$  as in (\*) above) then  $\eta \in A$  iff  $\eta \in B$ . Indeed, if  $\eta \in A$  (and  $\eta = \eta[G_\eta]$  where  $G_\eta$  is as in (\*)), by the definition of  $\{p_n : n < \omega\}$  and  $(\mathbf{t}_n : n < \omega)$ , there is some  $p_n \in G_\eta$  such that  $\mathbf{t}_n[G_\eta] = \text{true}$ , therefore  $\eta \in B_n \subseteq B$ . Similarly, if  $\eta \in B_n$  for some  $n$  such that  $\mathbf{t}_n[G_\eta] = \text{true}$ , then by the definitions of  $\{p_n : n < \omega\}$  and  $(\mathbf{t}_n : n < \omega)$ ,  $\eta \in A$ .  $\square$

**Conclusion 8:** A) The following theories are equiconsistent for  $i \in \{1, 2, 3\}$ :

1.  $Z^*C +$  "there is a limit cardinal  $> \aleph_0$ ".
2.  $Z^*C +$  "there is a strong limit cardinal  $> \aleph_0$ ".
- 3(i).  $Z^* +$  "every  $\Sigma_3^1$  set of reals is  $i$ -measurable".
- 4(i).  $Z^* +$  "every set of reals is  $i$ -measurable".

B) The following theories are equiconsistent for  $i \in \{1, 2, 3\}$ :

1.  $Z^*C +$  "there is a regular limit cardinal  $< \aleph_0$ ".
2.  $Z^* + C +$  "there is strongly inaccessible cardinal".
- 3(i).  $Z^* + DC +$  "every  $\Sigma_3^1$  set of reals is  $i$ -measurable".
- 4(i).  $Z^* + DC +$  "every set of reals is  $i$ -measurable".
- 5(i).  $Z^* + AC_{\aleph_0} +$  "every set of reals is  $i$ -measurable".  $\square$

### A remark on forcing over models of $Z_*$

In order to guarantee that the generic extensions in our proofs satisfy  $Z_*$ , we work in the context of models of  $Z_*$  of the form  $L$  or  $L[r]$  for some real  $r$ . In this context, we work with classes  $W$  of the following form: There is a formula  $\phi$  with parameters that defines the class, and there is a limit ordinal  $\nu < \omega^2$  such that  $\phi$  defines  $W \cap L_\alpha[r]$  in

$L_{\alpha+\nu}[r]$  when  $\alpha$  is a limit ordinal (recall that for every ordinal  $\alpha$ , the ordinal  $\alpha + \omega n$  exists).

Now, for a set forcing  $\mathbb{P}$  in a model of the above form, we define the class of  $\mathbb{P}$ -names as above. Therefore, for every limit ordinal  $\alpha$  we define the intersection of  $L_\alpha[r]$  with the class of names. For the names that we defined, we can prove the forcing theorem as usual and show that  $Z_*$  holds in the generic extension. In addition, note that when we force over  $L[r]$ , as  $L[r]$  has a well-ordering  $<_{L[r]}$  definable from  $r$ , we can use it to get a well-ordering of the generic extension, hence a model of  $Z_*C$ .

## Appendix: Can you take Solovay's inaccessible away? ([Sh176])

We now copy the definitions, theorems and proofs from [Sh176] that are relevant for understanding the above proofs.

**The following definitions are presented as step II in the above corresponding proof.**

**Definition 1.** 1. Let  $N_n$  be the set of pairs  $(t, m)$  such that:

- a.  $\emptyset \neq t \subseteq 2^{\leq n}$  is closed under initial segments, and for every  $\eta \in t \cap 2^{< n}$ , for some  $l$ ,  $\eta \hat{<} l > \in t$ .
- b.  $m : t \rightarrow \mathbb{Q}$  is a function such that  $m(<>) = \frac{1}{2}$ ,  $4^{lg(\eta)+1}m(\eta) \in \mathbb{N} \cap [1, 4^{lg(\eta)+1}2^{-lg(\eta)})$ , and for  $\eta \in t \cap 2^{< n}$ ,  $m(\eta) = \Sigma\{m(\eta \hat{<} l >) : \eta \hat{<} l > \in t\}$ .
2. Let  $N = \bigcup_{n < \omega} N_n$ , we call  $n$  the height of  $(t, m)$  for  $(t, m) \in N_n$  and denote it by  $ht(t, m)$ . If  $t' = t \cap 2^{\leq n}$ ,  $m' = m \upharpoonright t'$ , we let  $(t', m') = (t, m) \upharpoonright n$ . There is a natural tree structure on  $N$  defined by  $(t_0, m_0) \leq (t_1, m_1)$  if  $(t_0, m_0) = (t_1, m_1) \upharpoonright ht(t_0, m_0)$ .
3. A closed tree  $T \subseteq 2^{< \omega}$  satisfies  $(t, m)$  if  $T \cap 2^{\leq ht(t, m)} = t$  and for every  $\eta$ ,  $\mu(lim(T) \cap (2^\omega)_{[\eta]}) = m(\eta)$ .

**Definition 2.** 1.  $M_k$  is the set of pairs  $(t, m)$  such that for some  $n = ht(t, m)$  we have:

- a.  $\emptyset \neq t \subseteq 2^{\leq n}$  is closed under initial segments, and for  $\eta \in t \cap 2^{< n}$  there is  $l \in \{0, 1\}$  such that  $\eta \hat{<} l > \in t$ .
- b.  $m : t \rightarrow \mathbb{Q} \cap (0, 1)$  is a function such that  $m(<>) = \frac{1}{2}$ , and for  $\eta \in t \cap 2^{< n}$ ,  $m(\eta) = \Sigma\{m(\eta \hat{<} l >) : \eta \hat{<} l > \in t\}$ .
- c. We define  $r_l = lev_l(t, m)$  by induction on  $l$ :  $r_0 = 0$ ,  $r_{i+1}$  is the first  $r > r_i$  such that  $r \leq n$ , for every  $\eta \in 2^{\leq r} \cap t$ ,  $4^{r+1}m(\eta) \in \mathbb{N}$ , and for every  $\eta \in 2^r \cap t$ ,  $m(\eta) > 2^{-r}(1 - \frac{1}{\mu(l+1)})$ .

Now we demand that  $r_k$  is well defined and equals  $n$ .

2. Let  $M_{k,n} = \{(t, m) \in M_k : ht(t, m) = n\}$ ,  $M_{k,<n} = \bigcup_{l < n} M_{k,l}$ ,  $M_{*,<n} = \bigcup_{k < \omega} M_{k,<n}$ ,  $M = \bigcup_{k < \omega} M_k$ .
3. For  $(t, m) \in M_k$ , let  $rk(t, m) = k$ .

4. We define the order on  $M$  as we did for  $N$ .

**Definition 3:** A finite (full) system  $S$  consists of the following:

A. The common part: A finite subset  $W \subseteq \omega_1$  (the set  $W = \omega_1$ ) and a number  $n(1) < \omega$  ( $n(1) = \omega$ ) and a function  $h : [W]^2 \rightarrow n(1)$  such that if  $i_1 < i_2 < i_3$  belong to  $W$ , then  $h(i_1, i_2) \neq h(i_2, i_3)$ .

B. The red part:

a. For every  $(t, m) \in M_{*, \leq n(1)}$  there is a natural number  $\lambda(t, m)$ , and for every  $(t_1, m_1) \in N_{\lambda(t, m)}$  there is a member  $\rho(t_1, m_1, t, m) \in t \cap 2^{ht(t, m)}$ .

b. Let  $\{\eta_l : l < \omega\}$  be a fixed enumeration of  $2^{<\omega}$  such that  $lg(\eta_l) \leq l$ . For every  $(t, m) \in M_{k, \leq n(1)}$ ,  $l < k$ ,  $j < k$  and  $\xi \in W$ , there is a finite set  $A_{l, j}^{(t, m), \xi} \subseteq 2^{\leq \lambda(t, m)}$  such that  $\sum_{\nu \in A_{l, j}^{(t, m), \xi}} \frac{1}{2^{lg(\nu)}} < \frac{1}{2^{l+j}}$ .

c. For every  $(t, m) \in M_{k, \leq n(1)}$ ,  $\xi \in W$  and  $(t(0), m(0)) \in N_{\lambda(t, m)}$  there is a function  $f_{(t(0), m(0))}^{(t, m), \xi} : \{\eta_l : l < k\} \times k \rightarrow \omega$ .

d. Monotonicity for (a): If  $(t_0, m_0) < (t_1, m_1)$  (both in  $M_{*, \leq n(1)}$ ), then  $\lambda(t_0, m_0) < \lambda(t_1, m_1)$ . Moreover, if  $(t^0, m^0) < (t^1, m^1) \in N_{\lambda(t_1, m_1)}$ , then  $\rho(t^0, m^0, t_0, m_0) < \rho(t^1, m^1, t_1, m_1)$ .

e. Monotonicity for (b): If  $(t^0, m^0) < (t^1, m^1)$  (both in  $M_{*, \leq n(1)}$ ) and  $A_{l, j}^{(t^0, m^0), \xi}$  is defined, then  $A_{l, j}^{(t^0, m^0), \xi} = A_{l, j}^{(t^1, m^1), \xi}$ . Also  $f_{(t^0, m^0)}^{(t^0, m^0), \xi} \subseteq f_{(t^1, m^1)}^{(t^1, m^1), \xi}$  if  $(t_0, m_0) < (t_1, m_1) \in N_{\lambda(t^1, m^1)}$ .

f. The homogeneity consistency condition: If  $(t, m) \in M_{k, \leq n(1)}$ ,  $\xi < \zeta \in W$ ,  $h(\xi, \zeta) < ht(t, m)$ ,  $(t_1, m_1) \in N_{\lambda(t, m)}$  and  $\rho = \rho(t_1, m_1, t, m)$ , then:

1.  $\rho(h(\xi, \zeta)) = 0 (= red)$

or

2. For every  $l, j < k$ ,  $j \neq 0$  such that  $f_{(t_1, m_1)}^{(t, m), \zeta}(\eta_l, j) = f_{(t_1, m_1)}^{(t, m), \xi}(\eta_l, j)$  there is no perfect tree  $T \subseteq 2^{<\omega}$  which satisfies  $(t_1, m_1)$  and  $t_1^{\eta \leq}$  is disjoint to  $\bigcup_{\alpha < k} A_{\alpha, j}^{(t, m), \xi}$  and to  $\bigcup_{\alpha < k} A_{\alpha, j}^{(t, m), \zeta}$ .

C. The green part: It is defined similarly, only in (f)(1) we replace  $0 (= red)$  by  $1 (= green)$ .

**Definition 4.** The order between finite systems is defined naturally (for a given  $(t, m)$ ,  $\lambda(t, m)$ ,  $A_{l, j}^{(t, m), \xi}$ ,  $f_{(t(0), m(0))}^{(t, m), \xi}$  remain fixed,  $W$  and  $n(1)$  might become larger).

**The following claim corresponds to step III in the above proof.**

**Claim 5:** The family of finite systems satisfies the countable chain condition.

**Proof:** Let  $(S(\gamma) : \gamma < \omega_1)$  be a sequence of  $\omega_1$  conditions. By a delta-system argument, we may assume that for  $S(0)$  and  $S(1)$  we have:  $n := n(1)^{S(0)} = n(1)^{S(1)}$ ,

$\lambda^{S(0)} = \lambda^{S(1)}$ ,  $\rho^{S(0)} = \rho^{S(1)}$  and there is a bijection  $g : W^{S(0)} \rightarrow W^{S(1)}$  such that  $g$  is the identity on  $W^{S(0)} \cap W^{S(1)}$  and  $g$  maps  $S(0)$  onto  $S(1)$  in a natural way.

We shall define a common upper bound  $S$ . We let  $W^S : W^{S(0)} \cup W^{S(1)}$ ,  $n(1)^S = n+1$ . The function  $h^S$  is defined as follows: By the above claim, we may assume that  $h^{S(0)}$  agrees with  $h^{S(1)}$  on  $W^{S(0)} \cap W^{S(1)}$ .  $h^S$  will extend  $h^{S(0)} \cup h^{S(1)}$  as follows: If  $\xi < \zeta \in W^S$  and  $\xi \in W^{S(l)} \iff \zeta \notin W^{S(l)}$  ( $l = 0, 1$ ), then  $h^S(\xi, \zeta) = n$ . For each  $(t, m) \in M_{*, \leq n}$  we let  $\lambda(t, m)$ ,  $\rho(-, -, t, m)$  be as in  $S(0)$  and  $S(1)$ , and for  $\xi \in W^{S(l)}$ ,  $A_{l,j}^{(t,m),\xi}$  and  $f_{(t_1,m_1)}^{(t,m),\xi}$  are defined as in  $S(l)$ .

We shall now define the above information for  $(t, m) \in M_{*, \text{leqn}+1} \setminus M_{*, \leq n}$ . So let  $(t, m) \in M_{k+1, \leq n+1} \setminus M_{*, \leq n}$ , hence  $ht(t, m) = n+1$ . Clearly there is a unique  $(t(0), m(0)) < (t, m)$ ,  $(t(0), m(0)) \in M_{*, \leq n}$  ( $M_{k, \leq n}$ ). WLOG we shall concentrate on the red part. Define  $\lambda(t, m) = \lambda((t(0), m(0))) + |W^S| + (2k+1)$ . For every  $j \leq k$  define an independent family  $(A_{k,j}^{(t,m),\xi} : \xi \in W^S)$  of subsets of  $\{\nu : lg(\nu) = \lambda(t, m)\}$  such that  $\frac{|A_{k,j}^{(t,m),\xi}|}{2^{\lambda(t,m)}} = \frac{1}{2^{k+j+1}}$ .

Define  $f_{(t_1,m_1)}^{(t,m),\xi}(\eta_l, j)$  for  $(t_1, m_1) \in N_{\lambda(t,m)}$ ,  $j, l < k+1$  as follows:

1. If  $j, l < k$ ,  $\xi \in W^{S(l)}$ , let  $f_{(t_1,m_1)}^{(t,m),\xi}(\eta_l, j) = f_{(t_1,m_1) \upharpoonright \lambda(t_0,m_0)}^{(t_0,m_0),\xi}(\eta_l, j)$ .
2. If  $l = k$  or  $j = k$ , we think of  $f_{(t_1,m_1)}^{(t,m),\xi}(\eta_l, j)$  as a function of  $\xi$ , and we shall define it arbitrarily as an injective function to  $\omega$  (recalling that  $W^S$  is finite).

Defining  $\rho(t_1, m_1, t, m)$  for  $(t_1, m_1) \in N_{\lambda(t,m)}$ :

Let  $(t_0, m_0) := (t_1, m_1) \upharpoonright \lambda(t(0), m(0))$  (by monotonicity,  $\lambda(t(0), m(0)) < \lambda(t, m)$ ) and  $\rho_2 = \rho(t_0, m_0, t(0), m(0)) \in t(0)$ , so  $lg(\rho_2) = ht(t(0), m(0))$ . We shall find a proper extension  $\rho \in t$  of  $\rho_2$  that will satisfy definition 3(f). We shall consider the cases where 3(f)(2) fails, in each such case we need to guarantee that  $\rho(h(\xi, \zeta)) = 0$ . Now  $lg(\rho_2) = ht((t(0), m(0)))$ . Recall that  $(t(0), m(0)) \in M_{k, \leq n} \subseteq M_k$ , therefore,  $r_k$  in definition 2(c) exists and equals  $ht(t(0), m(0)) = \lambda(t(0), m(0))$ . By the definition of  $M_k$ ,  $m(0)(\rho_2) > 2^{-ht(t(0), m(0))}(1 - \frac{1}{\mu(k)})$ . By 2(b),  $m(0)(\rho_2) = \sum_{\rho_2 \leq \nu \in t \cap 2^{n+1}} m(0)(\nu)$ , now suppose that  $|\{\nu \in t \cap 2^{n+1} : \rho_2 \leq \nu\}| \leq 2^{(n+1)-lg(\rho_2)}(1 - \frac{1}{\mu(k)})$ , then  $m(0)(\rho_2) \leq 2^{-ht(t(0), m(0))}(1 - \frac{1}{\mu(k)})$  as  $m(0)(\nu) \leq 2^{-(n+1)}$  for every  $\rho_2 \leq \nu \in t \cap 2^{n+1}$ , which is a contradiction. Therefore,  $|\{\nu \in t \cap 2^{n+1} : \rho_2 \leq \nu\}| > 2^{(n+1)-lg(\rho_2)}(1 - \frac{1}{\mu(k)})$ . Therefore, if 3(f)(2) fails for less than  $\log(\mu(k))$  quadruples, then we can find  $\rho$  that satisfies the demands in 3(f) (suppose not, then for some  $c < \log(\mu(k))$ , there are  $c$  coordinates above  $lg(\rho_2)$  such that no extension of  $\rho_2$  in  $t$  of length  $n+1$  has 0 in those coordinates. There are  $\frac{2^{n+1}-lg(\rho_2)}{2^c}$  sequences with 0 in those coordinates, therefore, the number of extensions in  $t$  is at most  $2^{n+1}-lg(\rho_2)(1 - \frac{1}{2^c})$  which is a contradiction).

For a given pair  $(l, j)$  we want to count the number of  $\xi \in W^S$  such that  $t_1^{[n \leq]}$  is disjoint to  $\bigcup_{\alpha < k+1} A_{\alpha,j}^{(t,m),\xi}$ . Our goal is to show that 3(f)(2) fails for  $< \log(\mu(k))$

choices of  $(l, j, \xi, \zeta)$ . Now recall that  $lg(\eta_l) \leq l$ , and by the definition of  $N$ ,  $\frac{1}{4^{l+1}} \leq \frac{1}{4^{lg(\eta_l)+1}} \leq m_1(\eta_l)$ . As before,  $\frac{|t_1^{[\eta_l \leq 1]}|}{2^{\lambda(t(0), m(0)) - lg(\eta_l)}} > \frac{1}{4^{l+1}}$ . Recall that  $|\{\nu \in A_{k,j}^{(t,m),\xi} : \eta_l \leq \nu\}| = \frac{|A_{k,j}^{(t,m),\xi}|}{2^{lg(\eta_l)}}$ , therefore, if  $x$  is the number of sets  $A_{k,j}^{(t,m),\xi}$  that  $t_1^{[\eta_l \leq 1]}$  is disjoint to, then by a probabilistic argument,  $\frac{1}{4^{l+1}} < (1 - \frac{1}{2^{k+j+l}})^x$ .

As  $(1 - \frac{1}{2^{k+j+l}})^{2^{k+j+l}} < \frac{1}{e} < \frac{1}{2}$ , it follows that  $x < 2^{k+j+l}(2l+2)$ , so we have at most  $(2^{k+j+l}(2l+2))^2$  problematic pairs of  $(\xi, \zeta)$  for a given pair of  $(l, j)$ . Therefore, the number of problematic  $(l, j, \xi, \zeta)$  is at most  $\sum_{l,j < k} (2^{k+j+l}(2l+2))^2 < 2^{999k}$ , so by

letting  $\mu(k) = 2^{2^{999k}}$  we're done.  $\square$

**The following claim corresponds to step IV in the above proof.**

**Claim 6.** There is a full system in  $L[x_*]$ .

**Proof:** The existence of such a system can be described by a sentence  $\psi$  in  $\mathcal{L}_{\omega_1, \omega}$ , and by Keisler's completeness theorem it's absolute. By the previous claim, forcing with finite systems over  $L[x_*]$  preserves  $\aleph_1$ , hence we can get a full system in  $L[x_*]$ .

**The following claim corresponds to step V in the above proof.**

**Definition 7.** Fix a full system  $S$ . We define the formulas  $\phi_{rd}(x)$  and  $\phi'_{rd}(x)$  (and similarly,  $\phi_{gr}(x)$  and  $\phi'_{gr}(x)$ ) as follows:

1.  $\phi_{rd}(x)$  holds iff:
  - a. There is a tree  $T_0$  which is a poor man generic tree over  $L[x_*]$  (see see clause (A)(2) of step I in the above proof), so there is  $(n(k) : k < \omega)$  such that  $(t(k), m(k)) = (T_0 \upharpoonright 2^{< n(k)}, ms_T \upharpoonright 2^{< n(k)}) \in M_k$  (where for a closed tree  $T$ , the function  $ms_T$  is defined as  $ms_T(\nu) = \mu(\lim(T) \cap (2^\omega)^{[\eta \leq 1]})$ ).
  - b. There is a tree  $T_1$  which is a poor man generic tree over  $L[x_*, T_0]$  (see clause (A)(1) of step I in the above proof), so  $(t_n, m_n) = (T_1 \upharpoonright 2^{\leq n}, ms_{T_1} \upharpoonright 2^{\leq n}) \in N_n$  for every  $n < \omega$ .
  - c. For every  $k < \omega$ ,  $\rho_{rd}^S(t_{\lambda(t(k), m(k))}, m_{\lambda(t(k), m(k))}, t(k), m(k)) \leq x$ .
2.  $\phi'_{rd}(x)$  iff there is  $y$  such that  $\phi_{rd}(y)$  and  $x(n) = y(n)$  for  $n$  large enough.

**Claim 8.** There above formulas are  $\Sigma_3^1$ .

**Proof:** Being constructible from  $x_*$  is  $\Sigma_2^1$ , hence “for every  $G_\delta$  set  $B$  of measure 0,  $B \cap T = \emptyset$  or  $B$  is not constructible from  $x_*$ ” is  $\Pi_2^1$  and the conclusion follows.  $\square$

**The following claim corresponds to step VI in the above proof.**

**Claim 9.**  $\phi'_{rd}(x)$  and  $\phi'_{gr}(x)$  are contradictory.

**Proof:** Define a coloring of  $[\omega_1]^2$  by  $x(h(\xi, \zeta))$  for  $\xi, \zeta < \omega_1$ . If  $\phi_{rd}(x)$ , there are  $T_0, T_1$  and  $(n(k) : k < \omega)$  witnessing it. For  $j < \omega$ ,  $\eta_l \in T_1$  and  $\alpha < \omega$  let  $A_{j,l,n}$  be the set of  $\xi < \omega_1$  such that  $T_1^{\eta_l \leq 1}$  is disjoint to  $\bigcup_{l, k < \omega} C_{l,j}^{(t(k), m(k)), \xi}$  and  $f_{t_{\lambda(t(k), m(k))}, m_{\lambda(t(k), m(k))}}^{(t(k), m(k)), \xi}(\eta_l, j) = \alpha$  for large enough  $k$ . This is a partition of  $\omega_1$  to

---

countably many homogeneously red sets. Similarly, from  $\phi_{gr}(x)$  we get a partition of  $\omega_1$  to countably many homogeneously green sets, so we get a contradiction.

Now suppose that  $\phi_{rd}(x)$ ,  $\phi_{gr}(y)$  and  $x(n) = y(n)$  for  $n > n^*$ . There is a homogeneously red set  $A$  for  $x$  and a homogeneously green set  $B$  for  $y$  such that  $A \cap B$  is uncountable. There is an infinite set  $\{\xi_n : n < \omega\} \subseteq A \cap B$  such that  $h(\xi_{n_1}, \xi_{n_2}) < h(\xi_{n_2}, \xi_{n_3})$  has a fixed truth value for  $n_1 < n_2 < n_3$ . By definition 6(A),  $h(\xi_n, \xi_{n+1})$  is strictly increasing, hence it's  $> n^*$  for  $n$  large enough. Therefore, for  $n$  large enough,  $red = x(h(\xi_n, \xi_{n+1})) = y(h(\xi_n, \xi_{n+1})) = green$ , a contradiction.  $\square$

**Claim 13:**  $A_{rd} = \{x : \phi'_{rd}(x)\}$  is not of measure 0.

**Proof:** Suppose that  $b$  is a code for a  $G_\delta$  set of measure zero covering  $A_{rd}$ , then we get a poor man generic tree  $T_0$  over  $L[x_*, b]$  and a poor man generic tree  $T_1$  over  $L[x_*, b, T_0]$  (see step I in the above proof). Now let  $x \in \lim(T_0)$  such that  $T_0$  and  $T_1$  witness  $\phi'_{rd}(x)$ , then  $x$  is in no measure zero set coded in  $L[x_*, b]$ , contradicting the fact that  $x \in A_{rd}$  which is covered by the set coded by  $b$ .  $\square$

**Claim 14:**  $A_{rd}$  is not measurable.

**Proof:** By the previous claim, its measure is not zero. By the definition of  $\phi'_{rd}$ , the measure of  $\{x : \phi'_{rd}(x), \eta \leq x\}$  ( $\eta \in 2^{<\omega}$ ) is determined by  $lg(\eta)$ . Therefore  $A_{rd}$  has outer measure 1, and similarly for  $A_{gr}$ . As they're disjoint, we get a contradiction.  $\square$

## References

- [HwSh1067] Haim Horowitz and Saharon Shelah, Saccharinity with ccc, preprint
- [Sh176] Saharon Shelah, Can you take Solovay's inaccessible away? Israel J. Math. 48 (1984), no. 1, 1-47
- [So] Robert M. Solovay, A model of set theory in which every set of reals is Lebesgue measurable, AM 92 (1970), 1-56

(Haim Horowitz) Einstein Institute of Mathematics

Edmond J. Safra Campus,

The Hebrew University of Jerusalem.

Givat Ram, Jerusalem, 91904, Israel.

E-mail address: haim.horowitz@mail.huji.ac.il

(Saharon Shelah) Einstein Institute of Mathematics

Edmond J. Safra Campus,

---

The Hebrew University of Jerusalem.  
Givat Ram, Jerusalem, 91904, Israel.  
Department of Mathematics  
Hill Center - Busch Campus,  
Rutgers, The State University of New Jersey.  
110 Frelinghuysen road, Piscataway, NJ 08854-8019 USA  
E-mail address: [shelah@math.huji.ac.il](mailto:shelah@math.huji.ac.il)